PERFECT NUMBERS AND ABUNDANCY RATIO

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1. Introduction

Number theory is one of the oldest disciplines dating thousands of years back. Problems in number theory are remarkably easy to phrase and sometimes take centuries to prove. One of the unsolved problems involves perfect numbers, the concept of which dates back to the times of Euclid.

The purpose of this project is to give a brief but concise description of the developments in the study of perfect numbers.

A number n is called *perfect* iff $\sigma(n) = 2n$, where $\sigma(n) = \sum_{d|n} d$ is the sum of the positive divisors of n.

2. Even perfect numbers

Euclid proved that a number of the form $2^{p-1}(2^p-1)$, where both p and 2^p-1 are primes, is perfect. Euler proved the converse, i.e. that any even perfect number has the form specified by Euclid.

It should be noted that primes of the form 2^p-1 are called *Mersenne* primes and are usually denoted as M_p. It has been conjectured that there are infinitely many Mersenne primes (and so there are infinitely many even perfect numbers). As of the year 2000 only 38 Mersenne primes were known.

We can put the statements of Euler and Euclid into one theorem. Note, part of the proof of the theorem was taken from [Nar].

Theorem 1. *(Euclid, Euler) An even number is perfect iff it has the form $2^{p-1}M_p$ where M_p is the Mersenne prime.*

Proof. First let us prove Euclid's claim. Let $n=2^{p-1}M_p$ where M_p is a Mersenne prime. Since $\sigma(n)$ is a multiplicative function we obtain:

$$\begin{aligned}\sigma(n)&=\sigma(2^{p-1})\sigma(M_p)=(1+2+\cdots+2^{p-1})(1+M_p)=(2^p-1)(1+2^p-1)\\&=M_p\cdot 2^p=2n.\end{aligned}$$

So n is indeed perfect.

Now let us prove the converse of Euclid's statement. Since n is even, we can write it as $n=2^a m$, where m is odd. Then $2^{a+1}m=2n=\sigma(n)=(2^{a+1}-1)\sigma(m)$. Hence, 2^{a+1} must divide $\sigma(m)$ as it does not divide $2^{a+1}-1$. So $\frac{\sigma(m)}{2^{a+1}}\in\mathbb{Z}$, i.e. $\sigma(m)=2^{a+1}q$ where $q\in\mathbb{Z}$. So we have

$$m=(2^{a+1}-1)q \quad\text{and}\quad \sigma(m)=2^{a+1}q\,.$$

One can observe that $\sigma(m)=m+q$, so m and q are the only divisors of m and, therefore, m must be prime. Thus, $q=1$ and $1+m=\sigma(m)=2^{a+1}$. So we have $m=2^{a+1}-1$ is a prime. This implies that $a+1$ is also a prime, otherwise m would be composite. Setting $a+1=p$ gives the desired result. $\square$

3. Abundancy ratio

The search for perfect numbers gave rise to a rather interesting quantity called the *abundancy ratio*. It is defined as $\frac{\sigma(n)}{n}$ for a given integer n. Clearly, a number n is perfect iff its abundancy ratio is 2. Numbers for which this ratio is greater than 2 (less than 2) are called *abundant* (*deficient*) numbers.

Let us investigate certain properties of this ratio by stating and proving certain lemmas. First of all, it is quite clear to see that the abundancy ratio is a multiplicative function.

Lemma 1. $\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}$.

Proof. It is pretty much straight forward, namely: $\frac{\sigma(n)}{n} = \frac{1}{n}\sum_{d|n} d = \frac{1}{n}\sum_{d|n} \frac{n}{d} = \sum_{d|n} \frac{1}{d}$. □

Lemma 2. *If $m|n$ then $\frac{\sigma(m)}{m} \leq \frac{\sigma(n)}{n}$, with equality occuring iff $m = n$.*

Essentially this lemma says that any (nontrivial) multiple of a perfect number is abundant and every divisor of a perfect number is deficient.

Lemma 3. *The abundancy ratio takes on arbitrarily large values.*

Proof. Let's examine abundancy ratio of the number $n!$. Then, by Lemma 1, we have

$$\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq \sum_{i=1}^{n} \frac{1}{i},$$

and so, since the harmonic series diverges to infinity, we obtain the claim of this lemma. □

Lemma 4. *For any prime power p^a the following inequality holds*

$$1 < \frac{\sigma(p^a)}{p^a} < \frac{p}{p-1}.$$

Proof follows directly from the definition of σ function.

Since the abundancy ratio is a rational number, it would be nice to know the "distribution" of these ratios in the interval $[1, \infty)$. The next few results are related to it.

Theorem 2. *(Laatsch) The set of abundancy ratios $\frac{\sigma(n)}{n}$ for $n \geq 1$ is dense in the interval $[1, \infty)$.*

However, not all of the rationals from the interval $[1, \infty)$ are abundancy ratios of some integer. This is due to the following lemma (See [Wei]).

Lemma 5. *If* $\gcd(k, m) = 1$ *and* $m < k < \sigma(m)$*, then* $\frac{k}{m}$ *is not the abundancy ratio of any integer.*

Proof. Suppose $\frac{k}{m} = \frac{\sigma(n)}{n}$ for some integer n. Then $kn = m\sigma(n)$ meaning that $m|kn$, and so $m|n$ since m and k are coprime. However, by Lemma 2 we have

$$\frac{\sigma(m)}{m} \leq \frac{\sigma(n)}{n} = \frac{k}{m}$$

which yields $\sigma(m) \leq k$ - a contradiction to the fact that $k < \sigma(m)$. □

One can use previous lemma to establish another interesting theorem about the distribution of abundancy ratios.

Theorem 3. *The set of rationals that are not abundancy ratios is dense in* $[1, \infty)$*.*

The next theorem is a marvelous piece of work and demonstrates a very interesting link to odd perfect numbers. The proof was taken from [Wei].

Theorem 4. *If* $\frac{\sigma(n)}{n} = \frac{5}{3}$ *for some* n*, then* $5n$ *is an odd perfect number.*

Proof. For the given n we have $3\sigma(n) = 5n$, so $3|n$. If n is even, then $6|n$, and so by Lemma 2 we get $\frac{\sigma(n)}{n} \geq \frac{\sigma(6)}{6} = 2$, contradicting $\frac{\sigma(n)}{n} = \frac{5}{3}$. Thus n is odd, so $5n$ is also odd. Therefore, because $3\sigma(n) = 5n$, $\sigma(n)$ is odd as well. From the multiplicativity of $\sigma(n)$ it is easy to show that if n and $\sigma(n)$ are both odd, then n must be a square. Therefore $3^2|n$.

Does 5 divide n? If it does, then $3^2 \cdot 5|n$, and by Lemma 2 we get

$$\frac{\sigma(n)}{n} \geq \frac{\sigma(3^2 \cdot 5)}{3^2 \cdot 5} = \frac{26}{15} > \frac{5}{3},$$

contradicting $\frac{\sigma(n)}{n} = \frac{5}{3}$. Therefore, $\gcd(5, n) = 1$, so

$$\frac{\sigma(5n)}{5n} = \frac{\sigma(5)\sigma(n)}{5n} = \frac{6}{5} \cdot \frac{5}{3} = 2,$$

meaning that $5n$ is an odd perfect number. □

4. Odd perfect numbers

The situation is very different when we look at odd perfect numbers. Even their existence is in doubt. In fact, it is conjectured that odd perfect numbers do not exist. However, proving this conjecture turned out to be a very tough endeavour. We can see how rare odd perfect numbers really are by looking at the following theorem (See [BCR]).

Theorem 5. *(Brent, Cohen, te Riele) There is no odd perfect number less than* 10^{300}.

Euler was the first to actually decipher part of puzzle about odd perfect numbers, he derived the following theorem.

Theorem 6. *(Euler) Let n be an odd perfect number, then it has the form* $n = p^a \prod_{i=1}^{k} {q_i}^{2e_i}$, *where $p, q_1, q_2, \ldots, q_k$ are distinct primes and* $p \equiv a \equiv 1 \mod 4$.

Prime p in Euler's theorem is usually called the *special prime* and the factor p^a is called *Euler's factor*. From Euler's theorem we can deduce a rather simple, yet interesting corollary.

Corollary 1. *If n is an odd perfect number, then* $n \equiv 1 \mod 4$.

Applying Euler's theorem one can show the result describing the distribution of odd perfect numbers (See [Nar]).

Theorem 7. *If $P(x)$ is the number of odd perfect numbers not exceeding x, then* $P(x) \ll o(x)$.

No surprise there, since we already knew that odd perfect numbers are really rare, at least Theorem 7 quantifies their rarity.

It seems that so far there are two main types of theorems about odd perfect numbers: (1) theorems stating lower bounds on the size of divisors, and (2) theorems stating the minimum number of distinct primes. Obviously, these are not the only types of theorems, although they are very useful since they give us a rough framework to work in. We shall begin with the first kind of theorems and state the latest known results.

The next theorem states the size of the largest prime power factor in n (See [Coh]).

Theorem 8. *(Cohen) An odd perfect number must have a prime power divisor* p^a *which exceeds* 10^{30}.

Hagis and Cohen have shown the following theorem (See [HaC]).

Theorem 9. *(Hagis, Cohen) If* P *is the largest prime divisor of an odd perfect number, then* $P > 10^6$.

For the next two theorems see [Ian1] and [Ian2].

Theorem 10. *(Iannucci) If* S *is the second largest prime divisor of an odd perfect number, then* $S > 10^4$.

Theorem 11. *(Iannucci) If* T *is the third largest prime divisor of an odd perfect number, then* $T > 10^2$.

It should be duly noted that Theorems 8 - 11 have used the abundancy ratio extensively in their proofs, as well as making use of computer power.

The second kind of theorems are essentially based on a theorem of Dickson (See [Dic]).

Theorem 12. *(Dickson) For every* $k \geq 1$ *there can be at most a finite number of odd perfect numbers with* k *prime divisors.*

Recent improvement of Pomerance's upper bound is due to Heath-Brown (See [Hea]).

Theorem 13. *(Heath-Brown) If* n *is an odd perfect number with* k *distinct prime divisors, then*

$$n < 4^{4^k} \ .$$

Continuing in similar spirit, we shall state the latest results on the minimum number of prime factors of an odd perfect number (See [Hag1] and [Hag2]).

Theorem 14. *(Hagis) If* n *is an odd perfect number then it must have at least 8 distinct prime factors. Moreover, if* $3 \nmid n$ *then the minimum number of factors must be at least 11.*

Note that the first part of the previous theorem was independently proved by J. E. Z. Chein in his doctoral dissertation.

Certain other theorems deal with nonexistence of specific types of perfect numbers, however they are too numerous to list them all here. Instead we shall just give a taste of what they are all about by listing the simplest of cases.

Theorem 15. *(Pepin) An odd perfect number cannot be congruent to* $5 \mod 6$.

In contrast to Pepin's result, Touchard has shown that odd perfect numbers must satisfy certain congruences.

Theorem 16. *(Touchard) If* n *is an odd perfect number, then either* $n \equiv 1 \mod 12$ *or* $n \equiv 9 \mod 36$.

The next theorem has its value due to the restriction it places on the smallest prime divisor (See [Gru]).

Theorem 17. *(Grün) Let's say that an odd perfect number* n *has the form* $n = \prod_{i=1}^{k} {p_i}^{a_i}$ *where the primes* p_i *satisfy* $p_1 < p_2 < \cdots < p_k$. *Then* $p_1 < \frac{2}{3}k + 2$.

5. Conclusions

This area of research holds much promise. It seems that every so often we find more Mersenne primes, increase the search algorithms for odd perfect numbers, prove more theorems on the number of distinct primes of an odd perfect number etc. All of these tasks are computationally intensive, hopefully new computer technology will dramatically increase the results we obtain from these algorithms. Even though a lot of results about odd perfect numbers are known, it seems a far cry from a proof of (non)existence.

My personal feeling is that odd perfect numbers do not exist and that there are infinitely many Mersenne primes. Hopefully proving these conjectures will not take another millenia.

Due to the fact that Carleton Library did not have all the resources necessary for this project , the scope of the research has been somewhat limited. Nevertheless, I went through at least 40 papers (out of 100+), some of which did not have any new or relevant material and some of which were even absolutely wrong. Only the most recent results appear in this project.

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